ISSN 1541-308X

PHYSICS OF WAVE PHENOMENA

Vol. 12 No. 3 2004

Allerton Press, Inc. / New York

THE ALLERTON PRESS JOURNAL PROGRAM

PHYSICS OF WAVE PHENOMENA

Published under the combined auspices of the Wave Research Center, General Physics Institute, Russian Academy of Sciences and the United Physical Society of the Russian Federation

Published Quarterly

ALLERTON PRESS, INC.
18 West 27th Street, New York, NY 10001

PHYSICS OF WAVE PHENOMENA

Volume 12, Number 3, 2004

CONTENTS

Physics of Wave Phenomena
2004

Vol. 12. Number 3,
pp. 168–174

STABLE ALGORITHMS OF ADAPTIVE APPROXIMATION FOR ACOUSTIC SIGNAL DESCRIPTION BY ORTHOHONAL POLINOMIALS

A.K. Britenkov

Lobachevsky Nizhny Novgorod State University, 23 Gagarin Avenue, Nizhny Novgorod 603950, Russia

(E-mail: brak@rf.unn.ru)

A.N. Pankratov

Institute of Mathematical Problems of Biology, Russian Academy of Sciences, 4 Institutskaya Street, Pushchino, Moscow Oblast 142290, Russia

(E-mail: pan@impb.psn.ru)

(Received July 28, 2004)

We studied orthogonality loss during discretization of the definition domain of classical orthogonal polynomials of a continuous argument. An efficient algorithm for calculating high-order basis functions was proposed. Numerical experiments were carried out to estimate the accuracy of the proposed algorithm and Gaussian quadrature formulas. The problem of scale factor selection for adaptive approximation and study of various radiophysical signals was considered [1–4].

PACS: 02.60.Cb, Gf; 43.60.Lq, Mn

1. Introduction. Generalized Fourier series

Expansion of any function with respect to a set of orthogonal functions is a classical problem of functional analysis [5]. Numerical analysis includes several directions using orthogonal functions [6]. Generalized Fourier series are linear combinations

$$f(t) = \sum_n A_n \phi_n(t), \qquad (1)$$

where constants A_n are referred to as the expansion coefficients of function $f(t)$ over basis $\{\phi_n(t)\}$. The basis $\{\phi_n(t)\}$ satisfies the orthogonality condition

$$\begin{aligned} (\phi_i(t), \phi_j(t)) = 0 \quad \text{at} \quad i \neq j, \\ (\phi_i(t), \phi_j(t)) \neq 0 \quad \text{at} \quad i = j, \end{aligned} \qquad (2)$$

in the sense of the scalar product functional with weight $\rho(t)$ ($\rho(t) > 0$),

$$(x, y) = \int_a^b x(t)\, y(t)\, \rho(t)\, dt. \qquad (3)$$

The coefficients $\{\phi_i(t)\}$ of expansion in basis functions are determined from a set of linear equations derived by multiplying Eq. (1) by set $\{\phi_i(t)\}$,

$$\sum_{i=0}^{N} A_i(\phi_i, \phi_j) = (f, \phi_j), \quad j = 0, \ldots, N. \qquad (4)$$

The advantage of orthogonal bases lies in the fact that the expansion coefficients compose a diagonal (Graham) matrix and the explicit formulas

$$A_i = \frac{(f,\phi_i)}{(\phi_i,\phi_i)}, \quad i=0,\ldots,N, \tag{5}$$

are applicable to calculate expansion coefficients (1).

For any orthogonal expansion in series (1), the Bessel inequality (Lyapunov–Steklov equality at $N\to\infty$) characterizing completeness of set $\{\phi_i(t)\}$,

$$\|f(t)\|^2 \geq \sum_{n=0}^{N} A_n^2 \|\phi_n\|^2, \quad \|f(t)\| = \sqrt{(f(t),f(t))}. \tag{6}$$

For any functions f and g represented by series (1), the equality

$$(f,g) = \sum_{n=0}^{N} A_n B_n \|\phi_n\|^2 \tag{7}$$

is valid. This relation introduces the scalar product in space of expansion coefficients (isomorphism between initial functional and expansion coefficient spaces). The orthogonal basis $\{\phi_n(t)\}$ with weight $\rho(t)$ may be associated with the basis

$$\psi_n(t) = \sqrt{\rho(t)}\,\phi_n(t) \tag{8}$$

orthogonal with unit weight (isomorphism of Hilbert spaces).

Any function f is can be approximated with a specified accuracy ϵ at certain $N=N_\epsilon$, if the orthogonal set $\{\phi_i(t)\}$ is complete. Expansion coefficients (5) provide a minimum to the total error functional

$$\theta_N = \left\| f(t) - \sum_{i=0}^{N} A_i \phi_i(t) \right\|^2 = \|f(t)\|^2 - \sum_{i=0}^{N} A_i^2 \|\phi_i\|^2 \tag{9}$$

and are independent of N.

2. Discrete representation of basis functions

In problems of experimental data processing, the table representation

$$\widehat{f(t)} = \{f_1(t_1), f_2(t_2), \ldots, f_m(t_m)\}, \quad a \leq t_i \leq b, \tag{10}$$

of a signal under study, or an analog of a function of discrete argument, is initial. In this context, two problems arise: interpolation of function (10) of discrete argument and discretization of the scalar product of basis functions $\{\phi_n(t)\}$. The former problem is solved by data interpolation. The latter problem arises in numerical calculation of integrals, in particular, expansion coefficients A_n, and is subject to a stringent requirement of basis orthogonality conservation. For approximate calculation of integrals in a certain grid, orthogonality of $\{\phi_n(t)\}$ is in general violated. Expanding the set of nonorthogonal functions, coefficients A_n cannot be calculated using formulas (5).

A discrete analog of scalar product (3) in space of functions of discrete argument,

$$(x,y) = \int_a^b x(t)\,y(t)\,\rho(t)\,dt = \sum_{i=1}^{m} x(t_i)\,y(t_i)\,w_i, \tag{11}$$

defines the equality between the integral and sum only under certain conditions imposed on the integrand and certain selection of nodes t_i and weights w_i. Gaussian quadratures are characterized by the highest algebraic accuracy [7]. A special selection of nodes and weights (a total number of parameters is $2m$) allows fulfillment of the condition that the quadrature formula would be accurate for the integrand representing a polynomial of degree not higher than $2m-1$, multiplied by the weight function $\rho(t)$. It was proved [7] that nodes of quadrature formula (11) are zeros of an orthogonal polynomial of degree m, corresponding to the weight function $\rho(t)$. At such isomorphism between space of linear combinations (1) and space of functions of discrete argument in a specially chosen grid, grid function (10) defined in nodes of a polynomial of degree m is uniquely associated with a polynomial of degree $m-1$, multiplied by a root of weight function (8). Thus, Gaussian quadratures associate the orthogonal set of functions of a discrete argument with the orthogonal set of functions of a continuous argument.

3. Orthogonality violation during discretization

Let us consider orthogonality violation by the example of the Laguerre functions

$$l_n(t) = \sqrt{m}\exp\left(-\frac{mt}{2}\right)\sum_{k=0}^{n} C_n^{n-k}\frac{(-mt)^k}{k!}, \qquad (12)$$

where $m>0$ is the scale factor changing the effective length $l_n(t)$ [8]. Figure 1 shows the general view of the Laguerre function orthogonal in the semi-infinite interval $[0,\infty)$. We can see that this function is almost finite. Then the effective interval, in which the amplitude of function oscillations is of the same order of magnitude, linearly increases with n.

High-order quadrature formulas are constructed based on stable and effective calculation of basis functions. Methods for calculating nodes and weights for all the sets of classical orthogonal polynomials [8] are proposed in the Numerical Recipes package of procedures [9]. In this study, these procedures were modified using a more stable algorithm for calculating the Laguerre functions. In contrast to the Numerical Recipes package [9] containing procedures for polynomials, we constructed quadrature formulas for functions derived using Eq. (8). Zeros of polynomials and functions corresponding to them coincide, and weights of the Gaussian quadrature formula for polynomials and functions differ by the factor $\rho(t)$ only. Passage from polynomials to functions allows construction of high-order Gaussian quadrature formulas, since functions are limited.

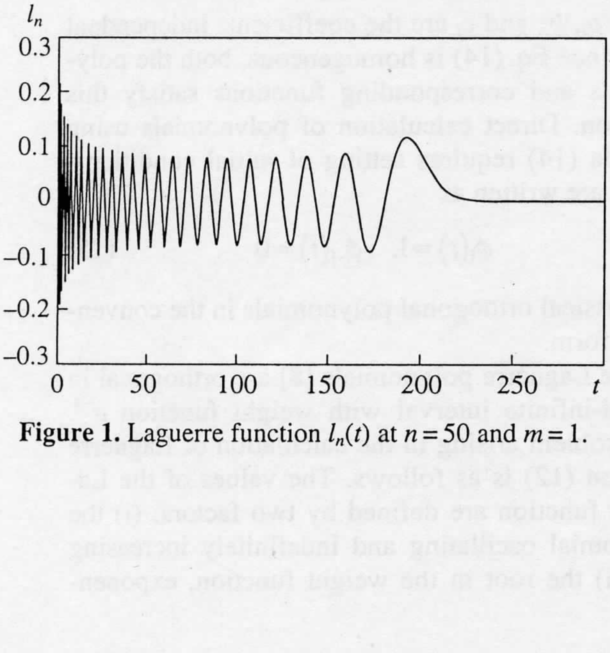

Figure 1. Laguerre function $l_n(t)$ at $n=50$ and $m=1$.

Figure 2 shows the calculated nodes and weights of the Gaussian quadrature formula for Laguerre functions. The nodes of the quadrature formula were determined numerically as zeros of the Laguerre orthogonal function. To determine and correct zeros most accurately and efficiently within the computer accuracy, three iterative methods were sequentially used: the Newton, regula falsi, and interval bisection methods. Newton iterations were efficient only for initial correction of the root, since, beginning from a certain iteration (mostly after the first one), they give rise to oscillations around the corrected root. When the root becomes limited from both sides, more efficient are iterations over a secant, without determination the function derivative. At the final stage, to correct the root in the last significant digits of the machine representation mantissa, the interval bisection method

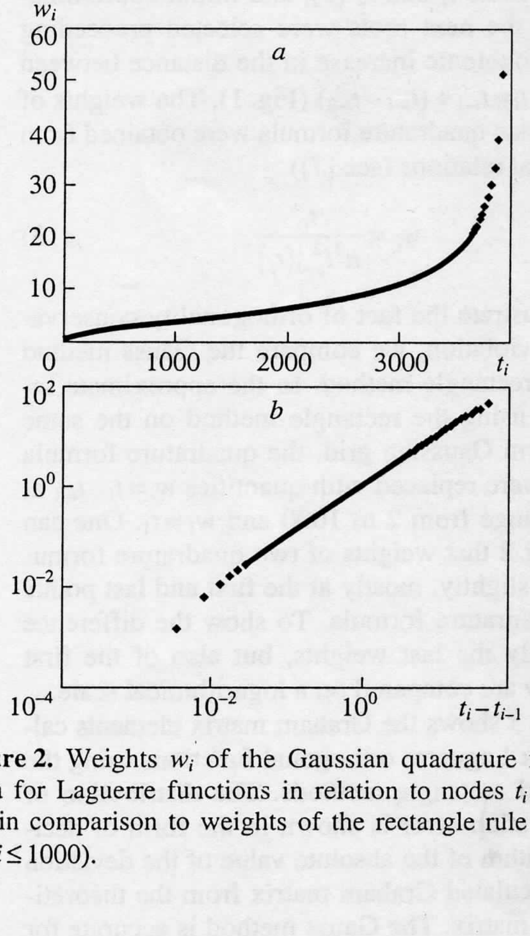

Figure 2. Weights w_i of the Gaussian quadrature formula for Laguerre functions in relation to nodes t_i (*a*) and in comparison to weights of the rectangle rule (*b*) $(0 \le i \le 1000)$.

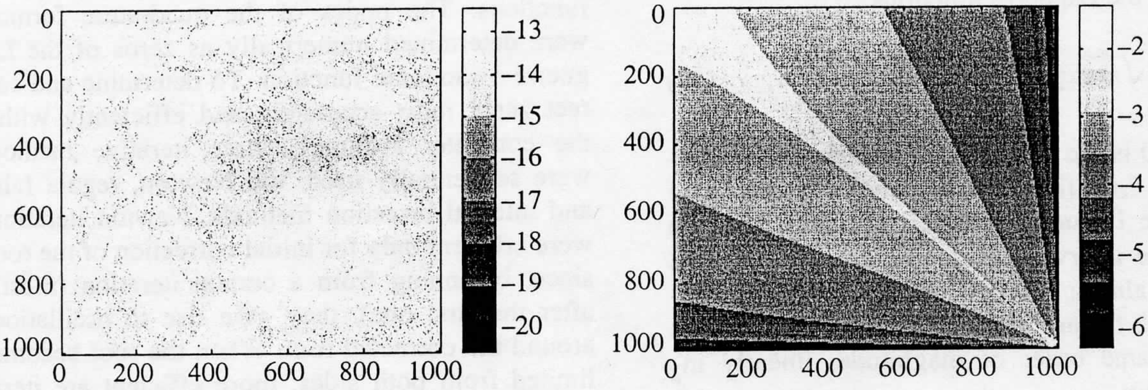

Figure 3. Error order in the calculation of Graham matrix elements (4) for Laguerre functions ($0 \le i < 1000$) using the Gauss (left) and rectangle (right) methods.

was used. There exist analytical initial approximations for roots t_1 and t_2 [9], and initial approximations for the next roots were selected proceeding from a monotonic increase in the distance between the roots $t_i \approx t_{i-1} + (t_{i-1} - t_{i-2})$ (Fig. 1). The weights of the Gaussian quadrature formula were obtained from the general relations (see [7])

$$w_i = \frac{t_i}{n^2 l_{n-1}^2(t_i)}. \tag{13}$$

To illustrate the fact of orthogonality conservation and violation, we compare the Gauss method with the rectangle method. In the approximate integration using the rectangle method on the same nonuniform Gaussian grid, the quadrature formula weights were replaced with quantities $w_i = t_i - t_{i-1}$ at i in the range from 2 to 1000 and $w_1 = t_1$. One can see in Fig. 2 that weights of two quadrature formulas differ slightly, mostly at the first and last points of the quadrature formula. To show the difference in not only the last weights, but also of the first ones, they are compared on a logarithmical scale.

Figure 3 shows the Graham matrix elements calculated for Laguerre orthogonal functions using the Gauss and rectangle methods. The distribution of the integration error is shown in the form of decimal logarithm of the absolute value of the deviation of the calculated Graham matrix from the theoretical (unit) matrix. The Gauss method is accurate for a chosen class of functions, a source of errors is cancellation of digits during approximate calculations. The rectangle method does not conserve orthogonality of the functions under consideration, and the error is systematic. This is indicated by the difference in the errors in the former and latter cases more than by ten orders of magnitude.

4. Construction of stable algorithms for calculating high-order orthogonal functions

Orthogonal polynomials satisfy the difference equations

$$\phi_{i+1}(t) = (a_i + b_i t)\,\phi_i(t) + c_i \phi_{i-1}(t), \tag{14}$$

where a_i, b_i, and c_i are the coefficients independent of t. Since Eq. (14) is homogeneous, both the polynomials and corresponding functions satisfy this equation. Direct calculation of polynomials using formula (14) requires setting of initial conditions, which are written as

$$\phi_1(t) = 1, \quad \phi_{-1}(t) = 0 \tag{15}$$

for classical orthogonal polynomials in the conventional form.

The Laguerre polynomials [8] are orthogonal in a semi-infinite interval with weight function e^{-t}. The problem arising in the calculation of Laguerre function (12) is as follows. The values of the Laguerre function are defined by two factors: (i) the polynomial oscillating and indefinitely increasing and (ii) the root in the weight function, exponen-

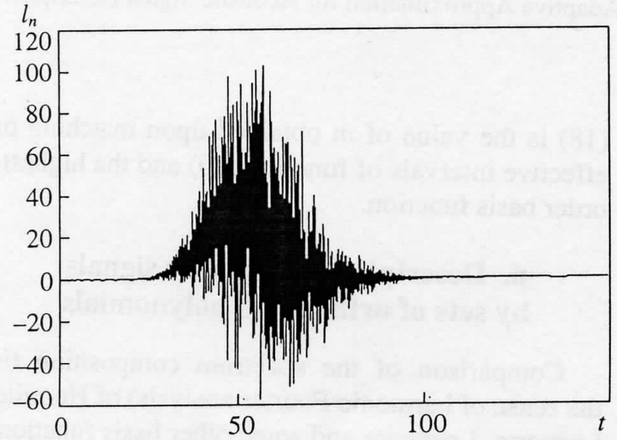

Figure 4. Laguerre function $l_n(t)$ calculated using Eq. (12) at $n=40$ and $m=1$.

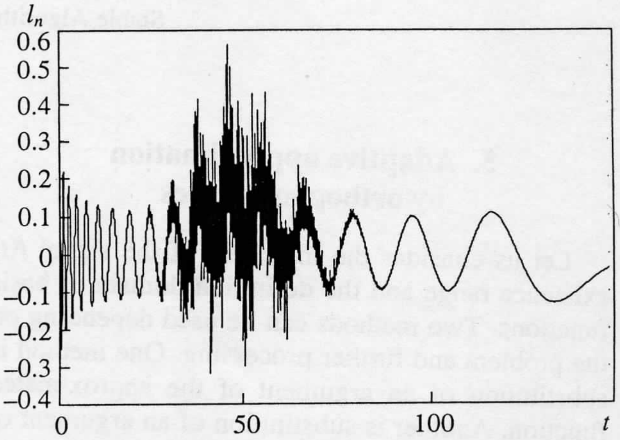

Figure 5. Laguerre function $l_n(t)$ calculated using Eq. (12) at $n=35$ and $m=1$.

tially decreasing as argument increases. At large enough t, these factors in machine representation result in overflow and disappearance of an order, respectively. However, their product, i.e., the Laguerre function, is a "good" value for machine representation. Observing the Laguerre functions beginning from the 33th order in the interval [0, 200], it is interesting to note that the results are repeated for direct algorithm of calculation by formula (12) using various softwares (Fig. 4). This is due to going over the range of allowed values during exponentiation (the exponent is 30 and above) of the argument t in the numerical interval from 100 to 160. In this case, the polynomial $l_n(t)$ contains a larger number of zeros than n (Fig. 4). It is more unexpected that the computing process becomes regular after a certain value of t (at $n=35$, $t=150$), and the $l_n(t)$ curve damps proportionally to e^{-t} (Fig. 5), as it can be expected from Eq. (12).

Calculation of the sought-for functions $l_n(t)$ using recurrent relations (14) and (15) also results in computing instability. The Laguerre functions l_n calculated by recurrent formulas (14), (15) and using Eq. (12) significantly differ at $n>40$. To solve these problems, the following procedure for calculating the Laguerre functions was used.

At each step of iteration cycle (14), the calculated values of $\phi_j(t)$ and $\phi_{j-1}(t)$ are multiplied by $\exp(-t/2n)$, where n is the Laguerre function order. For n cycles, the common factor will become a value of the factor required for determining the Laguerre functions, i.e., $\exp(-t/2)$. The calculation showed that such improvement of iterative process (14), (15) allows reliable calculation of the Laguerre functions above the 1000th order.

In this study, this idea is implemented in the following algorithm. At each step, normalization was carried out by variable 2^k (k is the order of binary representation of $\phi_j(t)$) rather than by a constant factor. In other words, the order of machine representation of $\phi_j(t)$ is nulled, while the orders of $\phi_{j+1}(t)$ and $\phi_{j-1}(t)$ are changed by k. The integer variable C contains a sum of changes in order k from the recurrent process beginning.

Finally, to obtain the value of the function $l_n(t)$ in n iteration by formula (14) with initial conditions (15), the result should be multiplied by the quantity

$$\exp(C \ln 2 - t/2). \tag{16}$$

Such improvement of the algorithm adds only integer operations; therefore, the function calculation efficiency does not decrease and the stability increases. This algorithm makes it possible to obtain the values of all the functions from the zeroth to $(n-1)$th order inclusively in n steps of recurrent process (14), which provides high processing speed of computing procedures [10]. No constraint for the order of calculated functions was detected (the algorithm was tested for the Laguerre and Hermite functions with $n=10000$).

5. Adaptive approximation by orthogonal series

Let us consider the matching of the signal $f(t)$ existence range and the definition domain of basis functions. Two methods can be used depending on the problem and further processing. One method is substitution of an argument of the approximated function. Another is substitution of an argument of basis functions. The latter method is more cumbersome, since it introduces the interval parameters into the basis parameters. However, this method is unique in the solution of inverse problems, when it is required to operate with an analytical form of expansion and the approximation interval parameters are unknown. This leads to construction of modified bases [6].

In the case of approximation by functions $l_n(t)$ in the semi-infinite interval $[0,\infty)$, coefficients A_i are functions of the scale factor m,

$$A_i(m) = \|\phi_i\|^{-2} \int_0^\infty f\left(\frac{t}{m}\right) \phi_i(t)\, \rho(t)\, dt. \qquad (17)$$

Expansion can be optimized by varying the scale coefficient m, i.e., the number N of terms of series (1) can be decreased at a specified accuracy ϵ. This is achieved by minimizing the root-mean-square error (9) as a function of the scale factor $\theta_N(m)$,

$$\theta_N(m_{\text{opt}}) = \min_{m>0} \theta_N(m). \qquad (18)$$

According to Eq. (9), the approximation error monotonically decreases with increasing N, although its dependence on m is more complex [6].

The effective duration of the Laguerre functions depends linearly on N. Let us consider the squared Laguerre function of the Nth order as the distribution density and determine an average radius of this distribution,

$$\int_0^\infty t\, L_N^2(t)\, e^{-t}\, dt = 2N+1. \qquad (19)$$

One of the possible version of initial choice of the scale factor for solving optimization problem (18) is the value of m obtained upon matching of effective intervals of function (10) and the highest-order basis function.

6. Description of acoustic signals by sets of orthogonal polynomials

Comparison of the spectrum composition (in the sense of harmonic Fourier analysis) of Hermite, Laguerre, Legendre and some other basis functions shows obvious similarity of various acoustic signals and some high-order basis functions. It is interesting to hear high-order orthogonal functions as sound signals. Many of them resemble natural sounds, hence, is convenient to synthesize sounds and distinguish acoustic signals using these functions. For example, Laguerre functions are ideal for sounds of bell ringing or gun shots. Hermite functions can be used to synthesize speech. The use of functions identical in form to eigenfunctions of a system under study makes it possible to simplify the mathematical model. Therefore, Laguerre functions are most optimum to describe transient processes as a response to a pulsed perturbation or such a function as

$$f(t) = 1 - \eta(t), \qquad (20)$$

where $\eta(t)$ is the Heaviside function.

We also carried out experiments on identification and analytical description of various acoustic signals, including speech ones, using the Chebyshev and Legendre functions [1].

Selection of an appropriate basis and its tuning by adaptive procedures such as (17) and (18) makes it possible to use 10–10 series terms for reliable identification.

7. Conclusion and discussion

The problems of numerical implementation of methods of expansion in classical orthogonal series, considered in this paper, showed that the use of Gaussian quadratures allows correct calculation of the integral of the scalar product, whereas the phenomenon of orthogonality violation arises when using other quadratures. The Graham matrix conventionally interpreted in computing mathematics as a measure of linear independence of a set of functions, was used as a measure of orthogonality in this study. The method proposed for calculating high-order Laguerre and Hermite functions can be applied to other orthogonal sets.

The urgency of the study of high-order orthogonal functions is confirmed in studies devoted to solution of differential equations [11], as well as study of random processes [12], development of the wavelet analysis [13] and generalized spectral-analytical method [6].

The authors are grateful to Prof. F.F. Dedus (Lomonosov State University) for his attention to this study and helpful remarks to the paper.

The work was supported by RFBR Projects 00-01-00417 and 01-02-16127, as well as the Federal Program of Support for Leading Scientific Schools, Project NSh-1641.2003.2.

References

1. Britenkov A.K. and Romanova I.V. Proc. Int. Youth Sci. Conf. “28th Gagorin Lectures”. Moscow: MATI, 2002. Vol. 2, p. 15 [in Russian].
2. Derome A.E. Modern NMR Techniques for Chemistry Research. Oxford: Pergamon, 1987.
3. Pankratov A.N. and Bystrova N.K. Proc. 7th Nat. School–Seminar “Wave Phenomena in Inhomogeneous Media”. Krasnovidovo, Moscow Oblast, 2000. Vol. 1, p. 16 [in Russian].
4. Artyukhin I.V. and Britenkov A.K. Proc. 10th All-Russian Conf. “Mathematical Methods of Image Recognition-10” (MMRO-10). Moscow: VTs RAN, 2001, p. 167 [in Russian].
5. Courant R. and Hilbert D. Methods of Mathematical Physics. Vol. 1. N.Y.–London: Interscience, 1953.
6. Dedus F.F., Makhortykh S.A., Ustinin M.N., and Dedus A.F. Generalized Spectral–Analytical Method for Processing of Information Arrays. Problems of Image Analysis and Recognition. Ed. F.F. Dedus. Moscow: Mashinostroenie, 1999 [in Russian].
7. Nikiforov A.F. and Uvarov V.B. Special Function of Mathematical Physics. Moscow: Nauka, 1984 [in Russian].
8. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Eds. M. Abramowitz and I. Stegun. N.Y.: Dover, 1972.
9. Press W.H., Teukolsky S.A., Vetterling W.T., and Flannery B.P. Numerical Recipes in C (The Art of Scientific Computing). 2nd Ed. Cambridge: Cambridge Univ., 1992.
10. Adaptive Approximation Program ADAP. http://impb.psn.ru/~pan/
11. Shen J. *SIAM J. Numer. Anal.* 2000, **38**(4), 1113.
12. Grafov B.P. and Grafova I.B. *Electrochem. Commun.* 2000, **2**, 386.
13. Astaf’eva N.M. *Phys.-Usp.* 1996, **39**(11), 1085.