Computational Mathematics and Mathematical Physics, Vol. 44, No. 12, 2004, pp. 2017–2023.
Translated from Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, Vol. 44, No. 12, 2004, pp. 2121–2127.

On the Implementation of Algebraic Operations on Orthogonal Function Series

A. N. Pankratov

Institute of Mathematical Problems of Biology, Russian Academy of Sciences, ul. Institutskaya 4, Pushchino, Moscow oblast, 142290 Russia

e-mail: pan@impb.psn.ru

Received December 12, 2003; in final form, July 1, 2004

Abstract—The implementation of algebraic operations on orthogonal series (namely, multiplication, division, and extracting the square root) is examined in the framework of the generalized spectral-analytical method. Two methods for defining multiplication by a function as an operator acting in the space of series coefficients are analyzed. It is shown that the commutative ring of functions from a finite-dimensional subspace of the Hilbert space L_2 is isomorphic to the ring of commuting symmetric matrices. This makes it possible to reduce the algebra of series to the algebra of matrices. Formulas are given for the orthogonal Chebyshev polynomials of the first kind.

Keywords: operator of multiplication by a function, orthogonal series, Chebyshev polynomials

1. INTRODUCTION

Algebraic operations are one type of analytic manipulations that can be performed with orthogonal series within the limits of the generalized spectral-analytic method [1]. According to the general methodology, the coefficients of a series obtained as a result of an analytic manipulation are expressed analytically in terms of the coefficients of the original series. In this paper, we give a detailed analysis of series multiplication. This analysis is required for the implementation of more complicated operations such as division and extracting the square root.

The algebraic operations under analysis do not impose any restrictions on the system of orthogonal functions to be used. Analytic properties of a basis determine the specific form that a transformation takes in the space of series coefficients but not the very possibility of performing this transformation. Therefore, the issue of performing such transformations is first examined purely theoretically for an arbitrary system of orthogonal functions. Then, by way of example, we derive formulas for the Chebyshev polynomials that can be directly applied in practice.

The choice of Chebyshev polynomials as one of the simplest bases from the viewpoint of performing the operations examined in this paper should be specially noted. For example, analytic operations on Chebyshev series are included in such a general package of numerical procedures as the Numerical Recipes [2]. This package includes implementations of series differentiation and integration as well as transformations between power and orthogonal series. Package [3], which is specially designed for Chebyshev polynomials, contains the multiplication procedure. This package also includes a procedure for solving boundary value problems for linear ordinary differential equations of order n. In [4], formulas are derived for multiplying functions constructed with the use of generalized Laguerre polynomials; then, these formulas are applied to solving a nonlinear differential equation.

The calculation of fractional powers of series is studied in [5, 6]. It is shown there that this calculation can be reduced to series multiplication and addition. The schemes proposed in those papers are iterative and are based on series addition and multiplication. The approach adopted in those papers is rather general and can be applied to objects of any nature for which addition and multiplication are defined, for example, to matrices and series. In this paper, we exploit a different approach based on the analysis of the space of series coefficients and the use of the properties of orthogonal series. The results obtained are valid for arbitrary orthogonal bases; for specific orthogonal systems, they can be further developed, as shown by using the orthogonal Chebyshev polynomials as an example.

2. SPACE OF SERIES COEFFICIENTS

Consider the orthogonal series of the form

$$a(x) = \sum_{k=0}^{N-1} a_k T_k(x), \tag{1}$$

where a_k are the coefficients and the functions $T_k(x)$ are orthogonal and normalized in the sense of the inner product generated by a positive weight function $\rho(x) > 0$:

$$(a(x), b(x)) = \int_{-1}^{1} a(x)b(x)\rho(x)dx. \tag{2}$$

The orthonormality conditions are as follows:

$$(T_i(x), T_j(x)) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{3}$$

In the space of series coefficients, the inner product (2) takes the form

$$(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{N-1} a_i b_i, \tag{4}$$

where **a** and **b** denote the vectors formed by the coefficients of the expansions of the functions $a(x)$ and $b(x)$, respectively. Note that, for infinite series ($N = \infty$), the equality of the inner products (2) and (4) follows from the completeness condition for the functions $T_k(x)$ (see [7]).

Consider the term-by-term product of two expansions:

$$c(x) = a(x)b(x) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i b_j T_i(x) T_j(x). \tag{5}$$

The product of two basis functions can, in general, be expanded in an infinite series in the same functions:

$$T_i(x)T_j(x) = \sum_{k=0}^{\infty} \delta_{ijk} T_k(x).$$

The coefficients of the resulting series are bilinear forms in the coefficients of the original series:

$$c_k = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \delta_{ijk} a_i b_j. \tag{6}$$

The coefficients of these forms are integrals of the triple products of the basis functions:

$$\delta_{ijk} = \int_{-1}^{1} T_i(x)T_j(x)T_k(x)\rho(x)dx. \tag{7}$$

The quantities δ_{ijk} are fundamental for deriving the rule by which the coefficients are transformed as a result of multiplying orthogonal series; they are determined by the chosen basis. Note that the value of δ_{ijk} is independent of the permutation of indices. For the classical orthogonal polynomials, $\delta_{ijk} = 0$ if the sum of the first two indices is less than the third index, i.e., if $i + j < k$.

We fix a function $b(x)$ and denote by the capital letter B the operator of multiplication by this function acting in the space of series coefficients. Then, expression (6) takes the form

$$c_k = \sum_{i=0}^{N-1} B_{ki} a_i, \tag{8}$$

or, in matrix form, $\mathbf{c} = B\mathbf{a}$, where the entries of the matrix associated with the operator B are as follows:

$$B_{ki} = \sum_{j=0}^{N-1} \delta_{ijk} b_j. \tag{9}$$

The coefficients of the square of a series can be found from (6):

$$c_k = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} a_i a_j \delta_{ijk}. \tag{10}$$

We examine two inverse problems: given a vector of coefficients c, find the coefficients in a. This corresponds to the division $a(x) = c(x)/b(x)$ in the case of the linear system (8) and to extracting the square root $a(x) = \sqrt{c(x)}$ in the case of the system of quadratic equations (10). Observe that, generally speaking, these systems are infinite. Answering the question of whether the finite subsystem of system (8) is solvable for $k = 0, 1, \ldots, N-1$ requires that the properties of matrix (9), generated by the operator of multiplication by a function, be examined.

Proposition 1. *The matrix of the operator of multiplication by a function is symmetric.*

Proof. This proposition is an immediate implication of definition (9) and the symmetry of the symbol δ_{ijk}. Note that the normalization of the basis where the operator is examined is an essential requirement. Proposition 1 is not valid for an orthogonal but not normalized basis. The proposition is proved.

Proposition 2. *The matrix of the operator of multiplication by a function is positive definite* (*i.e.*, $B > 0$) *if its generating function $b(x)$ is positive: $b(x) > 0$.*

Proof. Consider the following inner product:

$$(a(x)b(x), a(x)) = (B\mathbf{a}, \mathbf{a}) = \int_{-1}^{1} b(x)a^2(x)\rho(x)dx.$$

This inner product is first written in the space of series coefficients and then in the function space. If $b(x)$ preserves its sign on the entire integration interval, then the sign of the integral (hence, the sign of the infinite quadratic form $(B\mathbf{a}, \mathbf{a})$) is strictly definite whatever the choice of a. If the first N components of the vector $\mathbf{a}$ are nonzero while the others are zero, then $(B\mathbf{a}, \mathbf{a})$ becomes a finite quadratic form with a finite matrix B. The proposition is proved.

Theorem. *The eigenvalues of the matrix of the operator of multiplication by a function are bounded by the maximum and the minimum values of the generating function.*

Proof. The operator of multiplication by the constant function d has the diagonal matrix $D = d \times I$, where I is the identity matrix and all the diagonal entries of D are equal to d. Suppose that the function $b(x)$ generates the multiplication operator in accordance with (9) and has the maximum value $b_{\max}$ and the minimum value $b_{\min}$. Then, with the function $b(x) - b_{\min} \geq 0$, we associate the operator $(B - b_{\min} \times I) \geq 0$. It follows that all the eigenvalues of B cannot be less than $b_{\min}$. In a similar way, we prove that all the eigenvalues of B cannot be greater than $b_{\max}$. The theorem is proved.

The propositions and the theorem proved above give grounds for solving linear system (8). Proposition 2 and the Theorem make it possible to decide when the system is solvable and find its condition. The requirement that the sign of the function be permanent is consistent with the fact that division by a function that vanishes at some point of the interval of interest is inadmissible. Division by a function that is close to zero at some point results in a poor condition of the linear system (8). Another important conclusion is that system (8) can be solved by the iterative Seidel method, because the symmetry and the definiteness of a matrix are sufficient conditions for this method to converge for any initial approximation [8]. The Seidel method can also be used for solving the quadratic system (10).

3. THE SPACE OF GRID FUNCTIONS

In general, the space of square-integrable functions and its finite-dimensional subspaces are not closed with respect to multiplication of their elements. The results of multiplying of elements of a certain subspace belong, generally speaking, to the entire infinite-dimensional space or even may not belong to this Hilbert space. It follows that operators of multiplication by functions from a finite-dimensional subspace do not commute. The purpose of this section is to define multiplication in a finite-dimensional space in such a way as to make this space a commutative ring with respect to the operations of addition and multiplication.

Consider a finite-dimensional subspace of L_2 formed by linear combinations (1). This subspace is isomorphic to the space of the same dimension constituted by grid functions. For the classical orthogonal polynomials and functions, this isomorphism is provided by the Gaussian quadrature rule, which allows one to replace the integral in the definition of the inner product by the sum over the nonuniform grid formed by the zeros x_j of the orthogonal polynomial of degree N (see [9]):

$$(a(x), b(x)) = \int_{-1}^{1} a(x)b(x)\rho(x)dx = \sum_{j=1}^{N} a(x_j)b(x_j)w_j. \tag{11}$$

Here, the equality between the sum and the integral holds if the function $a(x)b(x)$ is a polynomial of degree not greater than $2N-1$; this is certainly true if $a(x)$ and $b(x)$ are polynomials of degree at most $N-1$.

In the space of grid functions, the multiplication operator has the simplest possible, namely, diagonal form. The eigenvalues of this operator are the values of the generating function at the grid points.

We construct the operator of multiplication by a function that acts in the space of series coefficients using the operator of multiplication by a function in the space of grid functions. Let M be the latter operator. Denote by T the matrix with the entries $T_{ij} = T_i(x_j)$, where x_j are the nodes of the Gaussian rule; T^* denotes the transpose of T, and W denotes the diagonal matrix with the weights w_j of the Gaussian rule on the main diagonal. Then, the desired operator can be written as the composition

$$T \times W \times M \times T^*. \tag{12}$$

This composition amounts to the successive (from right to left) application of the following transformations:

(1) The operator T^* ensures the calculation of the values of a series at the grid points, i.e., the transformation of the N-dimensional vector of series coefficients to the N-dimensional vector of values of the corresponding grid function;

(2) The operator M defines the multiplication of a function given by its values at the grid points by the function generating this operator;

(3) The operator $T \times W$ ensures the calculation of the series coefficients by the Gaussian rule, i.e., the inverse transformation from the space of grid functions to the space of series coefficients.

Consider the product of two operators of type (12):

$$TWM_1T^*TWM_2T^* = TWM_1(T^*TW)M_2T^* = TWM_1M_2T^*.$$

Since TW and T^* are mutually inverse operators (in view of the orthonormality of basis (3) and the Gaussian rule (12), we have $TWT^* = I$), the above equality proves the commutativity of multiplication for operators of this type.

Thus, we have a one-to-one correspondence between operators (12) acting in the space of series coefficients and those acting in the space of grid functions. Operator (12) provides an alternative definition of multiplication in a subspace of fixed dimension N of a Hilbert space. No knowledge of the analytic properties of the basis is required to construct this operator; one only needs to know the nodes and weights of the Gaussian rule. Therefore, this method for the construction of the operator of multiplication by a function can be called the method of quadrature discretization. Note that; among the classical orthogonal polynomials, the zeros are known analytically only for Chebyshev polynomials of the first and the second kind; hence, the proposed construction is, in general, a numerical method.

4. FORMULAS FOR THE CHEBYSHEV POLYNOMIALS

Chebyshev polynomials have the explicit trigonometric form

$$T_n(x) = \cos(n\arccos x), \quad -1 \le x \le 1.$$

The values of these polynomials can be calculated more efficiently by using the recursion

$$T_0 = 1, \quad T_1 = x, \quad T_{n+1}(x) = 2xT_n - T_{n-1}, \quad n \ge 1.$$

For a function of form (1), where $a(x)$ are Chebyshev polynomials, the coefficients can be found by the quadrature rule

$$a_j = \frac{2}{N}\sum_{k=1}^{N} a(x_k)T_j(x_k). \tag{13}$$

Here, the nodes are the zeros of the Chebyshev polynomial $T_N(x)$:

$$x_k = \cos\left(\frac{\pi(k-1/2)}{N}\right), \quad k = 1, 2, \ldots, N.$$

Furthermore, $j = 0, 1, \ldots, m-1$, and $m \le N$. Transformation (13) results in the expansion where the coefficient a_0 is multiplied by 1/2:

$$a(x) = \frac{a_0}{2} + \sum_{k=1}^{m-1} a_k T_k(x). \tag{14}$$

An immediate implication of the rule for the product of cosines

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$

is the following rule for the product of two Chebyshev polynomials:

$$T_iT_j = \frac{1}{2}T_{|i-j|} + \frac{1}{2}T_{i+j}. \tag{15}$$

Based on this rule and rule (5) for the term-by-term product of two series, we can propose the following algorithm for calculating the product. Each pair a_ib_j taken with the factor 1/2 yields an additive contribution to the two coefficients $c_{|i-j|}$ and c_{i+j} of the resulting series. To find the product of two series, each consisting of m terms, the required work is $m^2 + m$ multiplications and $2m^2$ additions.

To obtain explicit expressions for the coefficients of the product, we multiply two infinite series of type (14) term-by-term, apply formula (15), and combine the coefficients of identical polynomials:

$$c_k = \frac{1}{2}\sum_{j=0}^{k} a_jk_{k-j} + \frac{1}{2}\sum_{j=k+1}^{\infty}(a_jb_{j-k} + a_{j-k}b_j). \tag{16}$$

For instance, if $m = 5$, then the matrices of the bilinear forms (16) for $k = 0$ and $k = 2$ are as follows:

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \end{pmatrix}.$$

We rewrite the bilinear form (16) as a linear system with respect to the unknown vector of coefficients **a**:

$$c_k = \frac{1}{2}a_0b_k + \frac{1}{2}\sum_{j=1}^{\infty} a_j(b_{|j-k|} + b_{j+k}).$$

Consider the solution of this system by the Seidel method. The idea of this iterative method is to solve each equation of the system independently of the other equations by interpreting it as an equation in one unknown. The value found for the corresponding component of the vector is taken as a new approximation and is used in the other equations of the system. In this way, passing from one equation to the other, we refine the components of the vector of series coefficients.

We solve Eq. (16) with respect to the unknown a_k as follows:

$$a_k = \frac{1}{b_0}\left(2c_k - \sum_{j=0}^{k-1} a_j b_{k-j} - \sum_{j=k+1}^{m-1} (a_j b_{j-k} + a_{j-k} b_j)\right). \quad (17)$$

Note that the coefficient b_0 cannot vanish, because the function $b(x)$ has a constant sign. The initial approximation for Eqs. (17) can be chosen arbitrarily, since the convergence of the iterative process is ensured for any initial approximation.

In a similar way, we solve the system of quadratic equations:

$$a_0 = \sqrt{2\left(c_0 - \sum_{j=1}^{m-1} a_j^2\right)},$$
$$a_k = \frac{1}{a_0}\left(c_k - \frac{1}{2}\sum_{j=1}^{k-1} a_j a_{k-j} - \sum_{j=k+1}^{m-1} a_j a_{j-k}\right). \quad (18)$$

The convergence of the iterative process (18) is not justified theoretically. The initial approximations for Eqs. (18) should be chosen sufficiently close to the actual values to ensure that the coefficient a_0 does not vanish and the radicand in (18) is greater than zero. The proposed procedures were tested in numerical experiments. These procedures are most reasonable and efficient in problems where the operations of division and extracting the root are applied to a sequence of functions with slightly differing values, problems of continuation of functions by parameters being an example.

5. CONCLUSIONS

It is shown in this paper that one possible approach to the implementation of algebraic operations on orthogonal series is to reduce them to algebraic operations on matrices. This approach is based on defining the operator of multiplication by a function acting in the space of series coefficients. Its advantages are as follows:

(1) As soon as the operator of multiplication by a function is defined, the implementation of algebraic operations is independent of the basis;

(2) Being written in matrix form, the algorithms for transforming series are well suited for the implementation on vector and parallel computers;

(3) The multiplication operator along with the differentiation and integration operators can be used in solving differential and integral equations.

We showed that, theoretically, there are two nonequivalent methods for defining multiplication in a finite-dimensional subspace of a Hilbert space: the analytical method (9) and the numerical method (12), which is based on the quadrature discretization of a function.

ACKNOWLEDGMENTS

The author is indebted to F.F. Dedus and S.A. Makhortykh for statement of the problem and discussion of the results. This work was supported by the Russian Foundation for Basic Research, project no. 01-02-16127.

REFERENCES

1. F. F. Dedus, S. A. Makhortykh, M. N. Ustinin, and A. F. Dedus, *Generalized Spectral-Analytical Method for Data Processing: Image Analysis and Pattern Recognition* (Mashinostroenie, Moscow, 1999) [in Russian].
2. W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C* (Cambridge Univ. Press, Cambridge, 1992).

3. B. D' Aguanno, A. Nobile, and E. Roman, "CHPACK: A Package for the Manipulation of Chebyshev Approximations," Comput. Phys. Commun. **29** 361–374 (1983).

4. I. K. Khabibrakhmanov and D. Summers, "The Use of Generalized Laguerre Polynomials in Spectral Methods for Nonlinear Differential Equations," Comput. Math. Appl. **36** (2), 65–70 (1998).

5. R. Broucke, "Construction of Rational and Negative Powers of a Formal Series," Commun. ACM **14** (1), 32–35 (1971).

6. R. Broucke, "A446-Ten Subroutines for the Manipulation of Chebyshev Series," Commun. ACM **16** (4), 254–256 (1973).

7. R. Courant and D. Hilbert, *Methoden der matematischen Physik* 2 vols. (Springer, Berlin, 1931–1937; Gostekhteorizdat, Moscow, 1951).

8. N. S. Bakhvalov, N. P. Zhidkov, and G. M. Kobel'kov, *Numerical Methods* (Nauka, Moscow, 1987) [in Russian].

9. A. F. Nikiforov and V. B. Uvarov, *Special Functions in Mathematical Physics* (Nauka, Moscow, 1984) [in Russian].